Problem Set 4 – Relativity (G0I36A)

13/10/2020

1 Coordinate Transformations: Minkowski in Disguise!

Last week, we were introduced to the idea that we can start with in a particular coordinate frame, make coordinate transformations to a new set of coordinates, and then write down the metric spacetime in this new set of coordinates. That is, let's say that we have some set of coordinates x^μ for which the invariant line element ds takes the form

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \ , \tag{1.1}$$

where $g_{\mu\nu}$ is the metric tensor, and it can in principle be a function of the coordinates x^μ. Now, we consider making a coordinate transformation of the form

$$x^\mu \to x^{\mu'} \ , \tag{1.2}$$

for some new set of coordinates $x^{\mu'}$. In this new coordinate frame, the invariant line element ds will take the form

$$ds^2 = g_{\mu'\nu'} dx^{\mu'} dx^{\nu'} \ , \tag{1.3}$$

where $g_{\mu'\nu'}$ is the new metric tensor in this new coordinate frame, and it can in principle depend on the coordinates $x^{\mu'}$. The old metric tensor $g_{\mu\nu}$ and this new metric tensor $g_{\mu'\nu'}$ are related by the following equation:

$$g_{\mu'\nu'} = \frac{\partial x^\rho}{\partial x^{\mu'}} \frac{\partial x^\sigma}{\partial x^{\nu'}} g_{\rho\sigma} \ , \tag{1.4}$$

and thus we can determine the new metric tensor entirely simply by evaluating partial derivatives of the old coordinates x^μ with respect to the new coordinates $x^{\mu'}$.

In this section, we dive more into various coordinate transformations one can do that turn the usual Minkowski metric into something that looks much more non-trivial. Note that the underlying spacetime manifold will be the same; we are simply changing the set of coordinates we use to describe the manifold. Different coordinates will be useful for different purposes, and so getting the hang of coordinate transformations will be very useful for us.

1.) Prove equation (1.4). (I did this last week on the board, but try to do it on your own this time.) The easiest way is to start with (1.1) and then use the chain rule to put the line element in the form of (1.3) and then read off what $g_{\mu'\nu'}$ is.

2.) Consider the usual four-dimensional Minkowski spacetime with metric $ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu$. One simple coordinate transformation we can do is to pick a new set of coordinates $x^{\mu'} = (t', x', y', z')$ that are related to the original coordinates by both a shift and a scaling:

$$t' = \alpha_0 t + \beta_0 \ , \quad x' = \alpha_1 x + \beta_1, \quad \text{etc.} \ , \tag{1.5}$$

for some constants $\alpha_{0,1,2,3}$ and $\beta_{0,1,2,3}$. Derive the metric, as well as its inverse, in these new coordinates. You should find that the metric is independent of some of the parameters α and β. Physically, what is the reason behind this?

3.) Prove that the two-dimensional metric given by

$$ds^2 = dv^2 - v^2 du^2 \tag{1.6}$$

is equivalent to the two-dimensional Minkowski metric $ds^2 = -c^2dt^2 + dx^2$ by finding the coordinate transformations that bring us from one metric to the other.

4.) Define a set of rotating coordinates (t', x', y', z'), related to the original four-dimensional Minkowski coordinates (t, x, y, z) as follows:

$$\begin{aligned} t' &= t \ , \\ x' &= \sqrt{x^2+y^2}\ \cos(\phi - \omega t) \ , \\ y' &= \sqrt{x^2+y^2}\ \sin(\phi - \omega t) \ , \\ z' &= z \ , \end{aligned} \tag{1.7}$$

where $\tan\phi = y/x$. Find all components of the metric as well as its inverse. (Hint: you might find it easier to do this problem by first converting to cylindrical coordinates, i.e. writing down the relationship between the new coordinates (t', x', y', z') and the cylindrical coordinates (t, r, ϕ, z).)

2 Induced Metrics

Note: I'm about to write a lot of words in this section, but don't worry about the technical details too much. The core concepts are simple, and we will discuss induced metric and embeddings in great detail in class. The discussion below is really just to jog your memory later on.

Given a spacetime manifold $\mathcal{M}$, something we may be interested in is understanding the geometry on a subset $\Sigma \subset \mathcal{M}$ of the manifold. If this subset Σ is itself a manifold that has no self-intersections when placed inside of $\mathcal{M}$, we say that Σ is *embeddable* into $\mathcal{M}$. An equivalent way of phrasing this is that there exists an injective map f from Σ onto $\mathcal{M}$ that preserves the structure of $\mathcal{M}$.[1] You'll commonly see embeddings written as

$$f : \Sigma \hookrightarrow \mathcal{M} \ , \tag{2.1}$$

where the hooked arrow $\hookrightarrow$ is there to denote that f is more than a map, it's an embedding.

So, let's say that we start with a spacetime manifold $\mathcal{M}$, on which we have some coordinates x^μ for which the metric takes the form $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$. That is, we know how to measure the distance ds between any two points on $\mathcal{M}$ that are infinitesimally close to one another. If we have a submanifold (sometimes also referred to as a hypersurface) Σ that we can embed into $\mathcal{M}$, then we should also be able to measure the distance ds between points on Σ. After all, if we can measure distances on $\mathcal{M}$, we should also be able to measure distances on a subset of $\mathcal{M}$! Therefore if we denote the coordinates on Σ by y^α, then there must exist some metric tensor $\gamma_{\alpha\beta}$ such that the distance between two points on Σ is given by

$$ds^2|_\Sigma = \gamma_{\alpha\beta}dy^\alpha dy^\beta \ . \tag{2.2}$$

The metric on this submanifold is referred to as the *induced metric*.

[1] The precise definition of what it means for this map to be "structure-preserving" aren't going to be very important for us, so don't worry about this requirement too much. Just think of embeddings as smooth maps from a submanifold onto the larger manifold that are onto, i.e. no two points in Σ map to the same point of $\mathcal{M}$, and such that we can compute the induced metric on Σ.

An incredibly simple example of this is as follows. First, consider a four-dimensional Minkowski spacetime. Then, consider the submanifold (also called the hypersurface) defined by fixing $y = Y$, for some constant Y, while keeping t, x and z all free. Since y is fixed along this surface, there is no variation of the coordinate y, and so any dy terms in the line element ds^2 can be set to zero. The induced metric on this hypersurface is therefore simply

$$ds^2 = -c^2dt^2 + dx^2 + dz^2 \ , \tag{2.3}$$

which is easily recognized as the metric on a three-dimensional Minkowski spacetime with coordinates (ct, x, z). Thus, we have found the induced metric for $\mathbb{R}^{1,2} \subset \mathbb{R}^{1,3}$. [2]

Importantly, finding the induced metric on a submanifold is *not* the same thing as doing a coordinate transformation! Coordinate transformations are used to take you from one coordinate patch on a manifold to a different coordinate patch on the same manifold, whereas what we are doing here is describing an actual sub-manifold that is obtained by some coordinate restriction. This means that equation (1.4) does not apply in this situation. Instead, you'll have to use the chain rule together with the equations that define the submanifold to figure out what the line element ds on the submanifold looks like.

5.) First, start with the usual four-dimensional Minkowski spacetime. Then, consider a hypersurface in this spacetime defined by

$$t = T \ , \quad x^2 + y^2 = R^2 \ , \tag{2.4}$$

for some constants T and R, while z is unfixed. What is this hypersurface, and how many dimensions does it have? Compute the induced metric on this hypersurface. You may find it useful to first transform the starting Minkowski metric into cylindrical (aka polar) coordinates, and only then compute the induced metric.

6.) Consider another hypersurface in the four-dimensional spacetime, this time defined by

$$\begin{aligned} t &= T \ , \\ x &= (a + b\cos\theta)\cos\phi \ , \\ y &= (a + b\cos\theta)\sin\phi \ , \\ z &= b\sin\theta \ , \end{aligned} \tag{2.5}$$

where T, a, and b are all constants, while ϕ and θ are the coordinates on this hypersurface, with ranges $\phi \in [0, 2\pi)$ and $\theta \in [0, 2\pi)$. Compute the induced metric on this hypersurface.

7.) **Bonus:** Physically, what is this hypersurface? What do the parameters a and b correspond to?

3 Spheres and Loathing in Las Vegas

In this problem, consider a three-dimensional Euclidean space with metric $ds^2 = dx^2 + dy^2 + dz^2$ in the usual Cartesian coordinates.

8.) Please, double-check that you know how to compute the metric in spherical coordinates. It was on last week's problem set, but I really want you guys to be confident in this.

[2] The notation $\mathbb{R}^{n,m}$ is for a manifold with n time-like directions and m space-like directions, all of which take values on the real line. A d-dimensional Minkowski spacetime has a single time direction and $d-1$ spatial directions, and so we denote it by $\mathbb{R}^{1,d-1}$.

9.) Compute the components of the inverse metric.

10.) Suppose that a particle moves along a curve parameterized by

$$x(\lambda) = \cos\lambda\ , \quad y(\lambda) = \sin\lambda\ , \quad z(\lambda) = \lambda\ . \tag{3.1}$$

Express the path of the curve in spherical coordinates. Then, compute the tangent vector $V^\mu = \frac{dx^\mu}{d\lambda}$ along the curve in both the original Cartesian coordinates as well as spherical coordinates.

11.) Find the metric of a two-sphere (usually denoted by S^2) located at $r = R$, where R is a constant. Then, find the inverse metric.

12.) From there, argue that by fixing the coordinate θ on our S^2 to be constant, the resulting hypersurface is S^1. Compute the induced metric on this S^1.

13.) **Bonus:** prove that, for any value of $n \geq 1$, S^n is embeddable into $\mathbb{R}^{n+1}$. That is, prove there exists an injective map $f : S^n \hookrightarrow \mathbb{R}^{n+1}$ such that the induced metric on S^n is exactly the usual spherical metric.